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A Numerical Method Based on the NB1-Ball Polynomial for Solving a Class of Linear and Nonlinear Differential Equations

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ABSTRACT (10 PT)

In this article, NB1-Ball polynomials method for solving first and second orderordinary differential equation is proposed. Dealing with nonlinear and linear equations generated through matrix operation by simple form is the advantage of the suggested method. In order to show the performance of the proposed method, some real-life problems which include linear and nonlinear form of first and second order ordinary differential equations are introduced. The generated results confirm that the developed method outperform the existing method in terms of error.

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الخلاصة

في هذه المقالة ، تم اقتراح طريقة NB1-Ball متعددة الحدود لحل المعادلة التفاضلية العادية من الرتبتين الأولى والثانية. يعتبر التعامل مع المعادلات الخطية وغير الخطية الناتجة عن عملية المصفوفة بشكل بسيط ميزة الطريقة المقترحة. من أجل إظهار أداء الطريقة المقترحة ، تم تقديم بعض مشاكل الحياة الواقعية التي تشمل الشكل الخطي وغير الخطي للمعادلات التفاضلية العادية من الدرجة الأولى والثانية. تؤكد النتائج المتولدة أن الطريقة التي تم تطويرها تتفوق على الطريقة الحالية من حيث الخطأ.

1. INTRODUCTION

In this paper, we will present an efficient method for computing the numerical solution of differential equations (DEs). Problems of the type (9) and (10) have been considered by a vast number of scientific research fields, spanning from the chemical to the physical sciences and their applications to geophysics, reaction diffusion processes, and gas equilibrium, amongst a great number of other topics. As a result of the broad range of applications for problems of the kind under discussion, it is preferable to find a precise or approximate solution for the problem, which has been investigated by a large number of researchers. Nasab and Kilicman [1] used the wavelet analysis approach in order to solve linear and nonlinear initial (boundary) value problems. The Legendre operational matrix was used by Bataineh and Ishak Hashim [2] in order to come up with an approximation of the solution to two-point point boundary value issues. Bhatti [3] made use of the well-known Bernstein polynomial basis in order to find an approximate solution to the differential equation. Youseffi provided an approximate solution to the Bessel differential equation, and Yuzbasi also solved the fractional riccati type (DEs) [5], following the work of Bhatti, Pandey, and Kumar [4] and Isik and Sezer, who were able to get an analytic solution to the Lane-Emden type equations. In a recent paper, Yiming Chen used Bernstein polynomials in a similar way to turn up at the numerical solution to the variable order linear cable equation [6]. Rostamy also used a similar strategy to solve the backward inverse heat conduction problems [7], but he employed a modified operational matrix approach. Similar to the previous article, This paper likewise adopted the use of the NB1-Ball operational matrix to find solutions to issues involving linear and nonlinear starting (boundary) values. From the numerical answers produced, it is evident that there is commendable precision and reduced computing weight, as compared to the precise solution within a range of no more than 10 digits, just a few NB1-Ball polynomial basis functions are needed to get this approximative solution. This article is organised as follows: Section 2 discusses a review of Ball polynomials and NB1-Ball polynomials, as well as the conventional derivation of NB1-Ball polynomials and the differentiation of its operational matrix, while

Section 3 discusses applications of the operational matrix of the derivative. The numerical results are presented in Section 4, together with the precise solution, and the operational matrices validity, precision, and application are ultimately justified. Section 5 offers a succinct overview and conclusion.

2. REVIEW ON BALL POLYNOMIAL

The Ball polynomial was declared by A. A. Ball in his well-known aircraft design system CONSURF in [1]. It is described as a cubic polynomial and explained mathematically as:

$$(1-z)^2, 2z(1-z)^2, 2z^2(1-z), z^2, \quad 0 \le z \le 1. \tag{1}$$

In further research, several studies have discussed about Ball polynomial's high generalization and its properties. For instance, in the 1980s there were two different Ball polynomials of arbitrary degree are called Said-Ball and Wang-Ball [2, 3] and in 2003 there was another generalization of Ball polynomial called DP-Ball [4].

2.1. Nb1-Ball Polynomial Representation

Definition:

For any integer $n \ge 3$, the NB1 basis of degree n is defined as [5]

$$NB1_{i=0}^{n} = \begin{cases} \binom{\left\lfloor \frac{n}{2} \right\rfloor - 1 + i}{i} z^i (1-z)^{\left\lfloor \frac{n}{2} \right\rfloor}, & 0 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 2, \\ \binom{2\left\lfloor \frac{n}{2} \right\rfloor - 2}{\left\lfloor \frac{n}{2} \right\rfloor - 1} z^{\left\lfloor \frac{n}{2} \right\rfloor - 1}(1-z)^{\left\lfloor \frac{n}{2} \right\rfloor + 1}, & i = \left\lfloor \frac{n}{2} \right\rfloor - 1, \\ 2\binom{2\left\lfloor \frac{n}{2} \right\rfloor - 2}{\left\lfloor \frac{n}{2} \right\rfloor - 1} z^{\left\lceil \frac{n}{2} \right\rceil}(1-z)^{\left\lceil \frac{n}{2} \right\rceil}, & i = \left\lceil \frac{n}{2} \right\rceil, \\ NB1_{n-i}^{n}(1-z), & \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n. \end{cases} \tag{2}$$

Definition:

The NB1 basis function can be formulated in power basis form by [5]

$$\mathcal{C}^n(z) = \sum_{i=0}^{n} \sum_{j=0}^{n} b_{ij}\, z^j \tag{3}$$

where

$$b_{ij} = \begin{cases} (-1)^{(j-i)} \binom{\left\lfloor \frac{n}{2} \right\rfloor - 1 + i}{i} \binom{\left\lfloor \frac{n}{2} \right\rfloor}{j-i}, & \text{for } 0 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 2, \\ (-1)^{(j-i)} \binom{2i}{i} \binom{i+2}{j-i}, & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor - 1, \\ (-1)^{(j-i)} 2\binom{2i-2}{i-1} \binom{n-i}{j-i}, & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor, \\ (-1)^{(j-i)} 2\binom{2(n-i-1)}{n-i-1} \binom{n-i}{j-i}, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \\ (-1)^{(j-n+i)} \binom{2(n-i)}{n-i} \binom{n-i}{j-n+i-2}, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1, \\ (-1)^{(j-\left\lfloor \frac{n}{2} \right\rfloor)} \binom{\left\lfloor \frac{n}{2} \right\rfloor - 1 + n - i}{n-i} \binom{n-i}{j - \left\lfloor \frac{n}{2} \right\rfloor}, & \text{for } \left\lceil \frac{n}{2} \right\rceil + 2 \le i \le n. \end{cases} \tag{4}$$

Definition

The monomial matrix form for NB1-Ball can be specified as [6]

$$\mathcal{N} = \begin{bmatrix} g_{00} & g_{01} & \cdots & g_{0n} \\ g_{10} & g_{11} & \cdots & g_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ g_{n0} & g_{n1} & \cdots & g_{nn} \end{bmatrix}_{(n+1)\times(n+1)} \tag{5}$$

where g_{ij}, $i,j = 0,1,\cdots,n$ are given as (2).

In general, we approximate any function u(t) with the first m+1 NB1-Ball polynomials as:

$$y(z) \approx \sum_{i=0}^{m} c_i \mathcal{N}_i^m(z) = C^T\phi(z) = C^T\mathcal{N}T(z). \tag{6}$$

where $C^T = [c_0\ c_1\ c_2 \cdots c_m]$, $H(z) = [1\ \mathrm{z}\ \mathrm{z}^2 \cdots z^m]^T$ and $\mathcal{N}$ is the monomial matrix form was given in (5). The operational matrix of derivative of the NB1- Ball polynomials set $\psi(z)$ is given by $\frac{d\psi(z)}{dz} = D^{'(1)}\psi(z)$ is the $m+1$by $m+1$operational matrix of derivative define as

$$D^{'(1)} = \mathcal{N}\Lambda\mathcal{N}^{-1} \tag{7}$$

where $\mathcal{N}$is NB1-Ball monomial matrix form given in (5), and

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & m & 0 \end{bmatrix} \tag{8}$$

We can generalized Equation (8) as

$$D^{'(n)}\psi(z) = D^{'(n-1)}\left(D^{'(1)}\psi(z)\right) = \cdots = (D^{'(1)})^n\psi(z) = D^{'(n)}\psi(z), m = 1,2, \ldots$$

3. Applications of the Operational Matrix of Derivative

We present in this section the derivation of the method for solving differential equation of the form

$$q_0(z)u''(z) + q_1(z)u'(z) + q_2(z)(u(z))^n = g(z) \tag{9}$$

with initial conditions (ICs) or boundary conditions (BCs)

$$\begin{cases} u(0) = 1, \quad y'(0) = 0, \quad \text{or} \\ u(0) = \alpha_1, \quad\quad u(1) = \alpha_2. \end{cases} \tag{10}$$

where $q_j(z)$, $j = 0,1,2$ and $g(t)$ are given, while $u(z)$ is unknown. We can write the residual $\Re_n(z)$ as

$$\begin{aligned} \Re(z) \quad = \quad & q_0(z)C^TD^{'(2)}\psi(z) + q_1(t)C^TD^{'(1)}\psi(z) \\ & +q_2(z)(C^T\psi(z))^n - G^T\psi(z) \end{aligned} \tag{11}$$

where $G^T = [g_0, g_1, \cdots, g_m]$, To find the solution of $u(z)$ given in (10), we first collocate (12) at $m-1$ points. For suitable collection points, we use

$$z_i = \frac{1}{2} + \frac{1}{2}cos\left(\frac{(2i+1)\pi}{n}\right) \quad , i = 0,1,\cdots,m-1. \tag{12}$$

Theses equations together with (11) generate $m+1$ nonlinear equations which can be solved using Newton's iteration method. Consequently, $u(z)$ can be calculated.

4. NUMERICAL EXAMPLES

4.1. Example. 1

At the first we consider the example given in [6]

$$u''(z) + \frac{1}{z}u'(z) + u(z) = 4 - 9z + z^2 - z^3, \tag{13}$$

with BCs

$$u(0) = 0 \quad u(1) = 0. \tag{14}$$

Which has the exact solution is $u(z) = z^2 - z^3$.

To solve (13) and (14) we use our purposed method with $m = 3$. we apply (8) we have,

$$D^{'(1)} = \begin{bmatrix} -2 & -1 & -1 & 0 \\ 2 & -2 & -2 & 0 \\ 0 & 2 & 2 & -2 \\ 0 & 1 & 1 & 2 \end{bmatrix}, D^{'(2)} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ -8 & -2 & -2 & 4 \\ 4 & -2 & -2 & -8 \\ 2 & 2 & 2 & 2 \end{bmatrix}. \tag{15}$$

Therefore, using (13) for (14), we obtain

$$-\frac{55}{64}c_0-\frac{103}{128}c_1+\frac{115}{128}c_2+\frac{65}{64}c_3-\frac{115}{256} \tag{16}$$

$$\frac{67}{64}c_0+\frac{25}{128}c_1-\frac{501}{128}c_2+\frac{219}{64}c_3+\frac{501}{256} \tag{17}$$

Now we use the (BCs) we have

$$c_0=0, \quad c_3=0. \tag{18}$$

Solve Equations (17), (18) and (19) we get $c_0=0, \quad c_1=0, \quad c_2=\frac{1}{2}$ and $c_3=0.$ Thus

$$[u_3(z)]=c_0\mathcal{N}_0^3(z)+c_1\mathcal{N}_1^3(z)+c_2\mathcal{N}_2^3(z)+c_3\mathcal{N}_3^3(z)$$

$$=\begin{bmatrix}0 & 0 & \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}(z-1)^2\\ 2z(z-1)^2\\ -2z^2(z-1)\\ z^2\end{bmatrix}$$

$$=[z^2-z^3].$$

Which is the exact solution.

4.2. Example. 2

Consider the Bessel differential equation of order zero given in [7-10]

$$zu''(z)+u'(z)+zu(z)=0, \tag{19}$$

with the ICs

$$u(0)=1, \quad u'(0)=0. \tag{20}$$

The exact solution of this example is

$$J_0(z)=\sum_{q=0}^{\infty}\frac{(-1)^q}{(q!)^2}\left(\frac{z}{2}\right)^{2q}. \tag{21}$$

Here we see that $g(t)=0$ By the suggest method, we obtain the proximate solution when $m=12$ is

$$u_{12}=1.0-0.24999969t^2-0.00000712t^3+0.0156966t^4-0.0004143t^5+0.0010672128t^6$$
$$-0.003570235t^7+0.00565212t^8-0.00588660t^9+0.0038875t^{10}-0.001473t^{11}+0.0002435t^{12}.$$

The numerical results of our scheme together with two other [8, 9] are provided in Table 1

Table 1 Errors of the present method compared with results in ref [8, 9] for the Example 2.

t	PM m=12	Method of [9] for k=2, m=3,	Method of [8] for k=2, m=3
0.2	0	9.36e-05	6.01e-05
0.4	7.50e-11	2.78e-05	1.636e-04
0.6	3.24e-10	3.60e-05	1.636e-04
0.8	6.66e-09	2.695e-04	1.636e-04
1.0	1.66e-06	2.689e-04	1.636e-04

4.3. Example. 3

Consider the following ordinary differential equation [11]

$$u''(z) + zu'(z) + z^2u^3(z) = (2 + 6z^2)e^{(z^2)} + z^2e^{(3z^2)}, \quad (22)$$

Subject to IC

$$u(0) = 1, u'(0) = 0. \quad (23)$$

with the exact solution $u(t) = e^{t^2}$. We apply the above method when $m = 12$. Table 2 show the absolute error for Example 3.

Table 2 Errors of the present method compared with results in ref [11] for the Example 3.

t	Ref [17]	PM
0.000	0	0
0.010	0.2000000E-10	0.1683250E-10
0.020	0.2900000E-09	0.5781250E-10
0.030	0.2900000 E-09	0.1120996E-09
0.040	0.4450000 E-08	0.1725064E-09
0.050	0.1074000 E-07	0.2345360E-09
0.060	0.2207000 E-07	0.2956183E-09
0.070	0.4057000 E-07	0.3545185E-09
0.080	0.6872000 E-07	0.4108806E-09
0.090	0.1093000 E-06	0.4648881E-09
0.100	0.1654900 E-06	0.5170152E-09

4.4. Example. 4:

Consider the ordinary differential equation [11]

$$u''(z) + u(z)u'(z) = t\sin(2z^2) - 4z^2\sin(z^2) + 2\cos(z^2), z \in [0,1], \quad (24)$$

with ICs $u(0) = 0, u'(0) = 0$.
Where the exact solution is $u(z) = \sin(z^2)$. Table. 3 show the comparison the absolute error of our method with ref [11]

Table 3 Errors of the present method compared with results in ref [11] for the Example 4 with *m* = 12

t	Ref [11]	PM
0.0	0	0
0.1	3.074560E-7	7.249816E-9
0.2	1.058636E-5	1.483888E-8
0.3	5.114716E-5	2.254473E-8
0.4	1.331415E-4	3.054861E-8
0.5	2.420463E-4	4.135017E-8
0.6	3.299021E-4	5.825938E-8
0.7	3.231831E-4	9.301242E-8
0.8	1.540876E-4	1.668932E-7
0.9	1.870564E-4	3.322331E-7
1.9	6.088701E-4	3.322331E-7

4.5. Example. 5:

Consider the first order ode [11]

$$u'(z) - zu(z) + u^2(z) = e^{z^2} \quad (25)$$

subject to IC

$$u(0) = 1. \quad (26)$$

with the exact solution $u(z) = e^{\frac{z^2}{2}}$. The absolute error of Example .5 is presented in Table 4.

Table 4. Errors of the present method compared with results in ref [11] for the Example 5 with $m = 12$.

t	Ref [11]	PM
0.00	0	0
0.01	1.750000 E-7	2.610000E-11
0.02	6.400000 E-7	3.195700E-10
0.03	1.314000 E-6	5.506400E-10
0.04	2.123000 E-6	5.936400E-10

0.05	2.999000 E-6	4.440000E-10
0.06	3.883000 E-6	1.595400E-10
0.07	4.720000 E-6	1.791700E-10
0.08	5.463000 E-6	4.941400E-10
0.09	6.069000 E-6	7.240900E-10
0.1	6.501000 E-6	8.310500E-10

4.6. Example 6:

Finally Consider the following form of a singular Dirichlet-type boundary value problem on the interval [0, 1] [12]

$$u''(z) - \frac{1}{z}u'(z) + \frac{1}{z(1+z)}u(z) = -z^3, \tag{27}$$

with BCs

$$u(0) = 0, u(1) = 0. \tag{28}$$

where the exact Solution is

$$u(z) = \frac{1}{144(-1+2ln(2))}(14ln(z+1)t + 14ln(z+1) - 14z + 6z^2 - 12z^2 ln(2) - 2z^3 + 4z^3 ln(2) + z^4 - 2z^4 ln(2) + 9z^5 - 18z^5 ln(2)).$$

The absolute error in Table. 5 and in Figure. 1

Table 5 Errors of the present method compared with results in ref [12] for the Example 6 with $m = 11$.

t	Ref [12]	PM
0.2	1.88415721000000E–10	1.37812040000000E-10
0.4	7.13501861405898E–10	1.03644900000000E-10
0.6	8.20803253396388E–10	5.55406000000000E-11
0.8	5.53448662985227E–10	3.68906000000000E-11

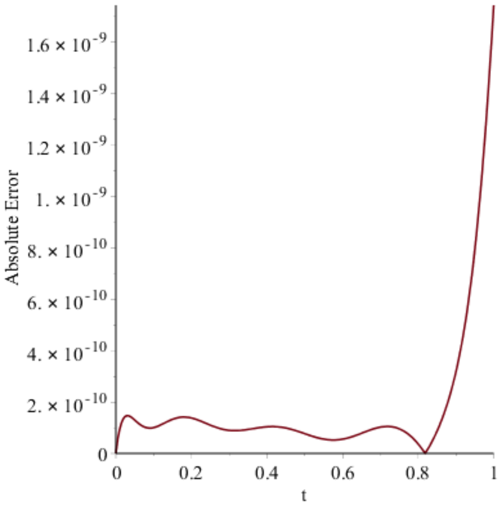

Figure 1. The absolute error for Example 6

5. CONCLUSION

In this work, the derivation of the new NB1-Ball polynomials method for solving first and second orders ODE is carried out. This new approach’s capacity to solve second orders ODE is its most significant advantage over those that have been previously proposed. The ability of the method is shown in its application to non-linear and linear first and second orders IVP and ICs of ODEs. The generated results approve the supremacy of new Said-Ball polynomials method over existing methods in terms of error as offered in tables 1-5.

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