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A numerical scheme for singularly perturbed parabolic convection-diffusion equation using Said-Ball Polynomial

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ABSTRACT

An This study introduces a numerical approach that converges uniformly for a convection-diffusion problem with singular perturbations. The collocation approach is used, and the derivative gets interpreted in the Caputo sense. Subsequently, a numerical scheme that converges uniformly is formulated using the Said-Ball collocation technique. Then, the primary issue may be simplified to a matrix equation that relates to a set of linear algebraic equations. Following the resolution of this system, the approximation of the provided problem's unknown Said-Ball coefficients is determined. The computational result is verified to be in agreement with the theoretical expectation and to be more precise than certain established numerical methods through numerical experimentation.

الخلاصة

تقدم هذه الدراسة منهجًا عدديًا يتقارب بانتظام لمسألة الحمل الحراري والانتشار مع الاضطرابات الشاذة. يُستخدم منهج التجميع، ويُفسر المشتق وفقًا لمفهوم كابوتو. بعد ذلك، تُصاغ خوارزمية عددية تتقارب بانتظام باستخدام تقنية التجميع سعيد-بول. ثم، يمكن تبسيط المسألة الأساسية إلى معادلة مصفوفية ترتبط بمجموعة من المعادلات الجبرية الخطية. بعد حل هذا النظام، يتم تحديد تقريب معاملات سعيد-بول المجهولة للمسألة المطروحة. وقد تم التحقق من صحة النتيجة الحسابية من خلال التجارب العددية، حيث تبين أنها تتوافق مع التوقعات النظرية وأنها أكثر دقة من بعض الطرق العددية المعروفة.

1. INTRODUCTION

The The second-order one-dimensional parabolic equation, as stated in [1-4], is the primary focus of this work.

$$u_\tau(\varsigma,\tau) - \varepsilon u_{\varsigma\varsigma}(\varsigma,\tau) + a(\varsigma)u_\varsigma(\varsigma,\tau) + b(\varsigma)u(\varsigma,\tau) = F(\varsigma,\tau), 0 \le \varsigma \le L, 0 \le \tau \le T. \quad (1)$$

where $a(\varsigma), b(\varsigma)$and $F(\varsigma,\tau)$ known real- valued functions and $\varepsilon < 1$is a known positive perturbation parameter that is generally taken to be close to zero. Equ. (1), known as the one- dimensional singularly perturbed convection-diffusion equation, will be considered under the initial condition (IC)

$$u(\varsigma,0) = g(\varsigma), 0 \le \varsigma \le L. \quad (2)$$

and the boundary conditions (BCs)

$$u(0,\tau) = h_0(\tau), u(L,\tau) = h_1(\tau), 0 \le \tau \le T, \quad (3)$$

where g, h_0 and h_1, as given by the initial and boundary conditions (2) and (3).

Consequently, various authors have developed an interest in acquiring its approximate solutions via the use of diverse numerical approaches. The convection–diffusion-reaction process consists of three distinct stages [5]. During the first stage, there is a transfer of convection and materials across different regions. In the second phase, there is a movement of diffusion and materials from an area with a high concentration to an area with a low concentration. The last stage is a process where decay, absorption, and the interaction of substances with other components take place.

Modeling difficulties in many scientific domains, including biology, physics, and engineering, may be rather complex due to the one-dimensional parabolic convection-diffusion equation, which is a partial differential equation [6–12]. Therefore, a number of scholars have set out to find numerical solutions to these difficulties by using various numerical techniques:

A Laguerre collocation approach was suggested by Gürbüz in order to resolve the 1D parabolic convection equation in [10]. A matrix-vector equation is transformed in this technique using the provided equation and conditions. Then, by employing collocation points, the Laguerre coefficients are derived from the solution of this matrix-vector equation. Lima et al. introduced a finite difference approach in [13] for both linear and nonlinear convection–diffusion–reaction models in order to get numerical results. The authors primarily concentrate on the examination of convergence, using errors and assessing the accuracy of the procedure. The authors in [14] presented an optimum q-homotopy analysis approach for obtaining an approximate solution to the convection-diffusion problem. Additionally, the convection-diffusion-reaction has been addressed using a number of different approaches, including the following: the homotopy perturbation method [15], the finite element method [16], the Runga Kutta method [17], the Bessel collocation method [2], the weighted finite difference [18], a hybrid approximation scheme [4], and the uniform convergent numerical method [19]. The Said-Ball collocation technique is used in this investigation, where it is the first time to be used to solve singularly perturbed parabolic convection-diffusion equation.

The paper is structured as follows: The already mentioned Said-Ball polynomial is discussed in Section 2. The paper illustrates the numerical scheme in Section 3. Section 4 of the paper provides a detailed explanation of a method called residual correction, which aims to enhance an existing solution. This method can also be utilized to estimate the error of the solution. In Section 5, two numerical examples are examined to exemplify the process of residual correction and to make comparisons with other methods. Section 6 contains the final remarks regarding the paper.

2. Said-Ball polynomials (SBP)

In this section, we will examine how the SBP may be utilized to create the operational matrix used to solve the 2nd order one-dimensional parabolic convection–diffusion equation under consideration. SBP is one of two generalized Ball polynomials (Said-Ball and Wang-Ball) of indeterminate degree established in the '80s [20, 21], both of which have the hallmark property of strong generalization among Ball polynomials. To be more specific, the Ball polynomial was first described in [21, 22], which defines a cubic polynomial as:

$$(1-\varsigma)^2, 2\varsigma(1-\varsigma)^2, 2\varsigma^2(1-\varsigma), \varsigma^2 \qquad (4)$$

according to the degree's parity, the SBP basis function of degree r, indicated by $S_k^r(\varsigma)$, is defined [23-27]. That is, when r is odd, $S_k^r(\varsigma)$is defined as

$$S_k^r(\varsigma) = \begin{cases} \binom{\frac{r-1}{2}+k}{k} \varsigma^k (1-\varsigma)^{\frac{r-1}{2}+1} & ,\text{for } 0 \le k \le \frac{r-1}{2}, \\ \binom{\frac{r-1}{2}+r-k}{r-k} \varsigma^{\frac{r-1}{2}+1} (1-\varsigma)^{r-k} & ,\text{for } \frac{r-1}{2}+1 \le k \le r. \end{cases}$$

when r is odd and

$$S_k^r(\varsigma) = \begin{cases} \binom{2^{-1}r+k}{k} \varsigma^k (1-\varsigma)^{2^{-1}r+1} & ,\text{for } 0 \le k \le 2^{-1}r+1, \\ \binom{r}{2^{-1}m} \varsigma^{2^{-1}r} (1-\varsigma)^{2^{-1}r} & ,\text{for } k{=}2^{-1}r, \\ \binom{2^{-1}r+r-k}{r-k} \varsigma^{2^{-1}r+1} (1-\varsigma)^{r-k} & ,\text{for } 2^{-1}r \le k \le r. \end{cases}$$

when r is even.

We can write the Said-Ball curve of degree r, denoted by$S_k^r(\varsigma)$, with $m+1$control points, denoted by $\{v_k\}_{k=0}^r$, can be written in terms of the power basis as follows [28]

$$S(\varsigma) = \sum_{k=0}^{r} \sum_{l=0}^{r} v_k \, m_{k,l} \varsigma^l, 0 \le \varsigma \le 1 \qquad (6)$$

where

$$m_{k,l}=\begin{cases}(-1)^{(l-k)}\binom{k+\left\lfloor\frac{r}{2}\right\rfloor}{k}\binom{\left\lfloor\frac{r}{2}\right\rfloor+1}{l-k}, & \text{for } 0\le k\le\left\lfloor\frac{r}{2}\right\rfloor,\\ (-1)^{(l-k)}\binom{r}{k}\binom{k}{l-k}, & \text{for } k=\frac{r}{2},\\ (-1)^{(l-\left\lfloor\frac{r}{2}\right\rfloor-k)}\binom{\left\lfloor\frac{r}{2}\right\rfloor+r-k}{r-k}\binom{r-k}{l-\left\lfloor\frac{r}{2}\right\rfloor-1}, & \text{for } \left\lfloor\frac{r}{2}\right\rfloor+1\le k\le r.\end{cases} \tag{7}$$

and $\lfloor\varsigma\rfloor$ and $\lceil\varsigma\rceil$ denote the greatest integer less than or equal to ς and the least integer greater than or equal to ς respectively

Definition:

The Said-Ball monomial matrix is [28]

$$M=\begin{bmatrix} m_{00} & m_{01} & \cdots & \cdots & m_{0N}\\ m_{10} & m_{11} & \cdots & \cdots & m_{1N}\\ \vdots & \vdots & \ddots & & \vdots\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ m_{N0} & m_{N1} & \cdots & \cdots & m_{NN}\end{bmatrix}_{(N+1)\times(N+1)} \tag{8}$$

where $m_{i,j}$ is given in Eq. (7)

3. METHOD OF SOLUTION

In this section, we will outline the procedure to be used to solve Equation (1) subject to initial and boundary conditions (2) and (3).

Firstly, we make the assumption that the solution in the truncated Said-Ball form

$$u(\varsigma,\tau)\cong u_N(\varsigma,\tau)=\sum_{m=0}^{N}\sum_{n=0}^{N}S_k^{m+1,n+1}(\varsigma,\tau)\,a_{mn} \tag{9}$$

where $S_{m+1,n+1}(\varsigma,\tau)=S_{m+1}(\varsigma)S_{n+1}(\tau)$ and $u_N(\varsigma,\tau)$ is the approximate solution of Eq. (1) $a_{m,n}, m, n=0,1,\dots,N$, are the unknown Said-Ball coefficients, N is chosen as any positive integer such that $N\ge 1$.

We can write

$$S(\tau)=X(\tau)M^T \tag{10}$$

Where $X(\tau)=[1\quad \tau\quad \tau^2\quad \cdots\quad \tau^N]$ and M given in Eq. (8). Then, by replacing the expression (10) into (9), we obtain the following matrix relations:

$$u_N(\varsigma,\tau)=X(\varsigma)M^T\bar{X}(\tau)\overline{M^T}A \tag{11}$$

where

$$\bar{X}(\tau)=I_N\otimes X(\tau),\overline{M^T}(\tau)=I_N\otimes M^T,$$
$$A=[a_{0,0}\quad a_{0,1}\quad \cdots\quad a_{0,N}\quad \cdots\quad a_{N,0}\quad a_{N,1}\quad \cdots\quad a_{N,N}]^T$$

On the other hand, the relation between the matrix $X(\tau)$ and its derivatives $X'(\tau)$ and $X''(\tau)$ are

$$X'(\tau)=X(\tau)\Lambda,\ X''(\tau)=X(\tau)\Lambda^2 \tag{12}$$

where

$$\Lambda=\begin{cases} i & ,j=i+1.\\ 0 & \text{, otherwise.}\end{cases} \tag{13}$$

Next, we arrange the matrix relations of the derivatives $u_\tau, u_{\varsigma\varsigma}$ and u_ς by using equations (10) - (12) in the following manner.

$$\begin{aligned} u_t(\varsigma,\tau)&=X(\varsigma)M^T\bar{X}(\tau)\bar{\Lambda}\overline{M^T}A,\\ u_\varsigma(\varsigma,\tau)&=X(\varsigma)\Lambda M^T\bar{X}(\tau)\overline{M^T}A,\\ u_{\varsigma\varsigma}(\varsigma,\tau)&=X(\varsigma)\Lambda^2M^T\bar{X}(\tau)\overline{M^T}A,\end{aligned} \tag{14}$$

By substituting the relations (14) into Eq. (1) we have the fundamental matrix form for Eq. (1):

$$\begin{aligned}&\Big\{X(\varsigma)M^T\bar{X}(\tau)\bar{\Lambda}\overline{M^T}-\varepsilon X(\varsigma)\Lambda^2M^T\bar{X}(\tau)\overline{M^T}\\ &+a(\varsigma)X(\varsigma)\Lambda M^T\bar{X}(\tau)\overline{M^T}+b(\varsigma)X(\varsigma)M^T\bar{X}(\tau)\overline{M^T}\Big\}A=F(\varsigma,\tau),0\le\varsigma\le L,0\le\tau\le T.\end{aligned} \tag{15}$$

or shortly

$WA = F$or $[W; F]$

where

$$W = X(\varsigma)M^T\bar{X}(\tau)\bar{\Lambda}\overline{M^T} - \varepsilon X(\varsigma)\Lambda^2 M^T\bar{X}(\tau)\overline{M^T} + a(\varsigma)X(\varsigma)\Lambda M^T\bar{X}(\tau)\overline{M^T} + b(\varsigma)X(\varsigma)M^T\bar{X}(\tau)\overline{M^T}$$

By putting the collocation points, for $\varsigma \in [0, L], \tau \in [0, T]$

$$\varsigma_i = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{i\pi}{N+1}\right), \tau_j = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{j\pi}{N+1}\right), i, j = 0,1, \dots, N. \quad (16)$$

into Eq. (15), then we have

$$W = [W_1 \quad W_2 \quad \cdots \quad W_N]^T,$$

$$W_i = [W(\varsigma_i, \tau_0) \quad W(\varsigma_i, \tau_1) \quad \cdots \quad W(\varsigma_i, \tau_N)]^T$$

$$G = [G_1 \quad G_2 \quad \cdots \quad G_N]^T,$$

$$G_i = [G(\varsigma_i, \tau_0) \quad G(\varsigma_i, \tau_1) \quad \cdots \quad G(\varsigma_i, \tau_N)]^T, i = 0,1, \cdots, N.$$

By replacing the relationship (16) in equations (2)-(3), we get the matrix representation.

$$u(\varsigma, 0) = X(\varsigma_i)M^T\bar{X}(0)\overline{M^T}A = g(\varsigma_i)$$

for the initial condition (2) and

$$u(0, \tau) = X(0)M^T\bar{X}(\tau_i)\overline{M^T}A = h_0(\tau_i),$$

$$u(L, \tau) = X(L)M^T\bar{X}(\tau_i)\overline{M^T}A = h_1(\tau_i)$$

for the boundary conditions (3), where $i = 0,1, \cdots, N$, or in short form

$$U_1A = G \text{ or } [U_1; G], U_2A = H_0 \text{ or } [U_2; H_0] \text{ and } U_3A = H_1 \text{ or } [U_3; H_1] \quad (17)$$

In order to get the solution to equation (1) given the conditions (2)-(3), an augmented matrix was created by substituting the row matrices (15) with the$(N + 1) \times (N + 1)$ rows from the matrix (17). This results in the formation of a new augmented matrix.

$$[\widetilde{W}; \tilde{G}] = \begin{bmatrix} W; F \\ U_1; G \\ U_2; H_0 \\ U_3; H_1 \end{bmatrix}$$

Then we solve the system $A = (\widetilde{W})^{-1}\tilde{G}$if $rank(\widetilde{W}) = rank(\widetilde{W}; \tilde{G}) = (N + 1)^2$and A is uniquely determined. So, the coefficients of the unknown Said-Ball polynomials are determined using this method. Therefore, the solution to $u_N(x, t)$ is approximately determined in the form of equation (9).

4. ERROR ANALYSIS

The estimated error for equation (1) is provided in this section; it enhances the accuracy of the solution for the Said-Ball polynomials. The resultant equation has to be satisfied approximately, that is, for $\varsigma = \varsigma_r, 0 \leq \varsigma_r \leq 1$ and $\tau = \tau_s, 0 \leq \tau_s \leq 1$.

$$E_N(\varsigma_r, \tau_s) = \left|u_\tau(\varsigma_r, \tau_s) - \varepsilon u_{\varsigma\varsigma}(\varsigma_r, \tau_s) + a(\varsigma_r)u_\varsigma(\varsigma_r, \tau_s) + b(\varsigma_r)u(\varsigma_r, \tau_s) - F(\varsigma_r, \tau_s)\right| \cong 0$$

Where $E_N(\varsigma_r, \tau_s) \leq 10^{-k_{rs}} = 10^{-k}$ (k is positive integer). If $\max 10^{-k_{rs}} = 10^{-k}$ is prescribed, then the truncation limit N is increased until the difference $E_N(\varsigma_r, \tau_s)$at each of the points becomes smaller than the prescribed 10^{-k}. On the other hand, we use absolute error (AE) for measuring errors. If $u_N(\varsigma, \tau)$ is an approximation to $u(\varsigma, \tau)$ the absolute error is $|e_N(\varsigma, \tau)| = |u(\varsigma, \tau) - u_N(\varsigma, \tau)|$. To facilitate the comparison of our findings with those of alternative approaches, we utilize L_2 norm L_∞ and norm, which are denoted as follows:

$$\|e_N(\varsigma,\tau)\|_2 = \left(\int_0^T \int_0^L \left(e_N(\varsigma,\tau)\right)^2 d\varsigma d\tau\right)^{1/2},$$
$$\|e_N(\varsigma,\tau)\|_\infty = \max_{(\varsigma,\tau)\in[0,L]\times[0,T]} |e_N(\varsigma,\tau)|.$$

5. NUMERICAL EXAMPLES

The procedure described in Section 3 is implemented on two illustrative problems in this section. Every necessary calculation has been performed using MATLAB R2021a

Example 1. The first example in our study is the following equation [1, 3, 4]

$$u_\tau - \varepsilon u_{\varsigma\varsigma} + (2\varsigma+1)u_\varsigma + \varsigma^2 u = \frac{e^{\varsigma+\tau}}{\varepsilon}(\varsigma^2 + 2\varsigma + 2 - \varepsilon), \quad (18)$$

with the initial condition

$$\boldsymbol{u(\varsigma,0) = \frac{e^{\varsigma}}{\varepsilon}, 0 \le \varsigma \le 1,} \quad \textbf{(19)}$$

and the boundary conditions

$$u(0,\tau) = \frac{e^{\tau}}{\varepsilon}, u(1,\tau) = \frac{e^{\tau+1}}{\varepsilon}, 0 \le \tau \le 1. \quad (20)$$

The exact solution of the present problem is $u(\varsigma, 0) = \frac{e^{\varsigma+\tau}}{\varepsilon}$.

We have utilized the approach outlined in Section 3 to examine Example 1, considering various options for N and employing multiple values for the perturbation parameter ε. Figure 1 shows the approximate solutions $u_6(\varsigma,\tau)$ for four different ε values.

To facilitate comparison with alternative collocation methods, we have computed the L_2 and L_∞ norms of the AE for N values ranging from 5 to 10. The values are presented in Table 1. While, Table 2 displays the AE for example 1, with $N = 10$ and $\varepsilon = 10^{-1}$, across various values of τ.

TABLE 1 Comparison of the L_∞error of the AE function $|e_N(\varsigma,\tau)|$for different values of N andin Example 1 ε

PM	N =5	N =6	N =7	N =8	N =9	N =10
ε =1/10	8.4771E-04	4.4025E-06	3.0556E-07	1.3021E-08	6.2679E-10	3.3103E-11
ε =1/100	8.4771E-04	5.2696E-05	1.3459E-06	3.7924E-08	1.0859E-09	3.0996E-11
ε =1/1000	8.4320E-03	5.2635E-04	1.2767E-05	3.8824E-07	9.7772E-09	3.5053E-10
ε =1/10000	8.4309E-02	5.2623E-03	1.2712E-04	3.8995E-06	9.7036E-08	3.4861E-09
Reff [3]	N =5	N =6	N =7	N =8	N =9	N =10
ε =1/10	1.9640E−3	1.0855E−4	8.6060E−6	1.1654E−7	1.2083E−9	2.3913E−10
ε =1/100	4.3049E−2	1.5669E−3	1.3818E−4	2.0306E−6	3.8459E−8	1.5497E−8
ε =1/1000	4.7793E−1	7.1433E−2	1.1717E−2	1.9467E−4	2.2718E−6	1.2584E−7
ε =1/10000	4.8544	9.8674E−1	1.6973E−1	1.1336E−2	8.2980E−5	5.1276E−6
Reff[13]	N =5	N =6	N =7	N =8	N =9	
ε =1/10	9.6181E−4	1.8000E−5	1.5525E−6	1.2692E−5	6.8182E−9	
ε =1/100	6.0181E−3	2.2000E−4	1.1333E−5	1.1429E−7	8.5000E−8	
ε =1/1000	6.3998E−2	2.1500E−3	1.1365E−4	1.3333E−6	9.2500E−7	
ε =1/10000	6.5455E−1	2.1500E−2	1.1500E−3	1.3429E−5	9.0000E−6	

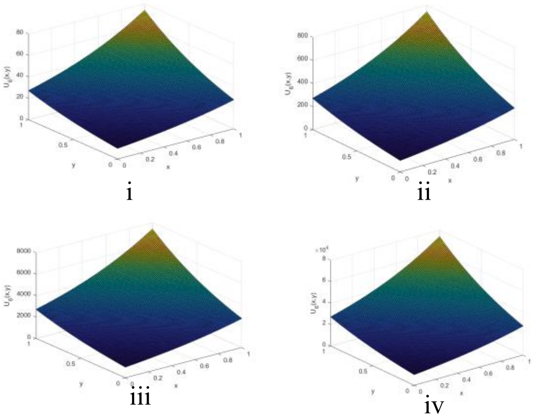

Figure 1. Approximate solutions of Example. 1 obtained with $N = 6$corresponding to i, $\varepsilon = 1/10$, ii, $\varepsilon = 1/100$, iii, $\varepsilon = 1/1000$ and iv, $\varepsilon = 1/10000$.

Table 2 Comparison the AE for example 1, with $N = 10$ and $\varepsilon = 10^{-1}$, across various values of τ.

ς_i	$\tau = 0.1$	$\tau = 0.3$	$\tau = 0.5$	$\tau = 0.9$
0.1	9.4378E-05	1.6155E-04	1.2113E-04	3.3651E-04
0.2	1.1199E-06	1.7145E-04	3.1997E-05	2.9740E-04
0.3	3.4494E-05	1.6925E-04	6.8859E-05	2.2835E-04
0.4	3.4143E-05	1.4561E-04	1.3862E-04	1.6038E-04
0.5	2.2180E-05	9.5216E-05	1.5400E-04	8.7212E-05
0.6	7.4082E-06	3.3329E-05	1.2800E-04	1.4748E-05
0.7	1.8610E-05	7.8109E-06	1.0204E-04	2.6218E-05
0.8	6.4979E-05	4.9280E-06	1.1082E-04	7.9679E-06
0.9	1.0471E-04	5.7055E-06	1.1704E-04	2.1253E-05

Example 2. Next, we will address the problem that was already analyzed in references [3, 4].

$$u_\tau - \varepsilon u_{\varsigma\varsigma} + (2 - \varsigma^2)u_\varsigma + \varsigma u = 10\tau^2 e^{-\tau}\varsigma(1 - \varsigma), \varsigma, \tau \in [0,1]. \quad (21)$$

Both the initial as well as the boundary conditions could be given by:

$$u(\varsigma, 0) = 0, \varsigma \in [0,1], \quad u(0, \tau) = u(1, \tau) = 0, \tau \in [0,1]. \quad (22)$$

Since the exact solution of this problem is not known, the residual function $R_N(\varsigma, \tau)$ to assess the accuracy of the approximate solutions will be utilized. Example 2 is the one to which the present scheme has been applied. In Fig. 2 illustrates the residual functions of the approximate solutions obtained with different Nvalues and for $\varepsilon = 2^{-4}$.

Furthermore, In figure 3, we have implemented the current technique on Example 2 using $N = 8$and the singular perturbation parameter values of $\varepsilon = 2, 4, 6$, and 8. However, the data in table 3 demonstrate that the current strategy produces outcomes that are similar to the other ways stated for this specific case. Finally, Table 4 presents the AE for example 2, considering different values of τ, $N = 7$, and $\varepsilon = 2^{-2}$.

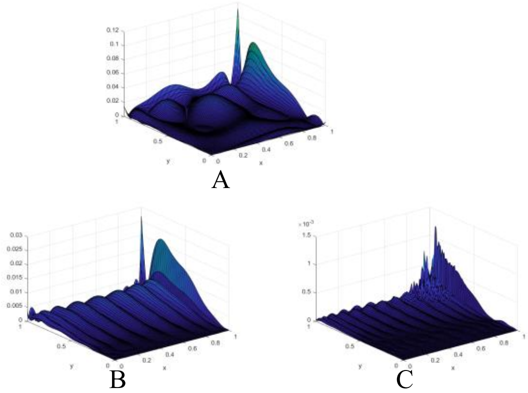

Fig 2. The residual functions of the approximate solutions for example 2, derived for A with N=6, B with N=10, and C with N=14, correspond to the selected perturbation parameter $\varepsilon = 2^{-4}$.

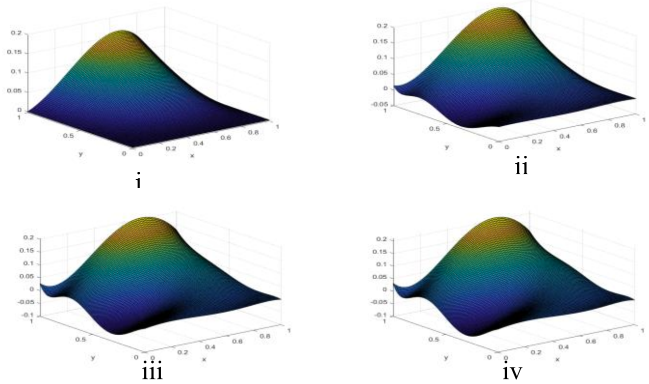

Fig. 3. Approximate solutions of Example 2 obtained with N=8 corresponding to i, $\varepsilon = 1/4$, ii, $\varepsilon = 1/16$, iii, $\varepsilon = 1/64$ and $\varepsilon = 1/256$.

TABLE 3. Comparison of the L_2error of the absolute error function $|e_N(\varsigma,\tau)|$for various values of $N$and εin Example 2

ε		2^{-2}	2^{-4}	2^{-6}	2^{-8}
PM	N=3	0. 15940E-3	0. 17052E-3	0. 17252E-3	0. 17377E-3
	N=4	0. 46406E-4	0. 82402E-4	0. 10558E-3	0. 11430E-3
Reff [3]	N=3	0. 1071E−3	0. 3357E−3	0. 8856E−3	0. 5429E−3
	N=4	0. 2723E−4	0. 2630E−3	0. 6464E−3	0. 4001E−3
Reff [2]	N=3	0. 1791E−3	0. 2454E−3	0. 4272E−3	0. 2909E−3
	N=4	0. 1090E−4	0. 1141E−3	0. 1187E−3	0. 8395E−2
Reff [29]	N=16	0. 2030E−3	0. 2810E−3	0. 3048E−2	0. 8395E−2
	N=32	0. 1113E−3	0. 1857E−3	0. 1275E−2	0. 4648E−2
Reff [30]	N=16	0. 26E−04	0. 115E−3	0.225E−3	0. 152E−3
	N=32	0. 9921E−5	0. 51E−4	0. 167E−3	0. 144E−3
Reff [31]	N=3	0. 1124E−3	0. 1678E−3	0. 3090E−3	0. 3574E−3
	N=4	0. 6320E−4	0. 8104E−4	0. 1522E−3	0. 1934E−3

Table 4. Comparison the AE for example 2 at $N = 7, \varepsilon = 2^{-2}$.

ς_i	$\tau = 0.1$	$\tau = 0.3$	$\tau = 0.5$	$\tau = 0.9$
0.1	1.8775E-05	2.9553E-05	2.3824E-05	2.7127E-04
0.2	5.3067E-05	6.3429E-05	2.0241E-05	6.2175E-04
0.3	1.4144E-05	2.1355E-05	3.7880E-05	1.9695E-04
0.4	6.7517E-05	5.0091E-05	2.3373E-05	5.5325E-04
0.5	6.9006E-05	3.4627E-05	7.4278E-05	4.6528E-04
0.6	2.8147E-05	4.3531E-05	6.4821E-06	3.2309E-04
0.7	1.0545E-04	6.8340E-05	1.2288E-04	6.9734E-04
0.8	2.2380E-05	1.0176E-05	6.5462E-05	2.7561E-05
0.9	1.2731E-04	5.6492E-05	2.0063E-04	6.9274E-04

6. Conclusions

This work presents a collocation technique that is built upon the Said-Ball approach. The method is designed to numerically solve convection-diffusion equations of parabolic type, which are often encountered in several engineering fields. The primary characteristic of the work being given is the need to solve an algebraic system of equations at each individual time step, as opposed to solving a global system produced in Said-Ball collocation techniques. The accuracy and efficiency of the suggested technique are shown by numerical tests, which are described in figures and tables. These results are compared with existing published schemes. The suggested approach can be expanded to include the fractional solutions of the singularly perturbed parabolic convection-diffusion equation.

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